Les Mathématiques dans l'Art

Mathematics in Art

What happens when a hammer hits a piano string?

Travail de Maturité

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monsieur David Taj

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Foreword

This work was made in LaTeX. The template was provided by the school. The pictures that are not sourced in the webography were either made by myself with Mathematica, Excel or hand drawn and modified with photo editing software. The spectrogram was made with Audacity.

Smith Harry

Introduction

The first instrument is thought to date back about 40'000 years[1]. Throughout history, music has been a great source of inspiration and emotion for humans. According to Martin Luther: "Beautiful music is the art of the prophets that can calm the agitations of the soul; it is one of the most magnificent and delightful presents God has given us". One of the first known records of musical analysis is attributed to Pythagoras. He defined a scale (which is a system in which notes are associated with certain frequencies) based on mathematical properties, so the link between mathematics and music has a long history. However, he was far from having a thorough understanding of music theory and furthermore the Pythagorean scale is unpleasant to listen to! The connections between music and maths are very much still a field of enquiry.

I will be exploring what happens when a hammer hits a piano string. Since the age of six, I have played the piano and I now compose music for the piano and on my computer. The piano is an extremely complex instrument with many moving parts and it has a complicated physical structure. There are many kinds of pianos, including grand, upright and electric pianos. The word piano is an abbreviation of piano-forte. It was named as such because the piano hammer can hit the string at different velocities, which allows for more expressive playing (going from piano to forte, which means soft to loud). Most pianos have 88 keys, meaning it is one of the instruments with the widest ranges, with over seven full octaves. It is a versatile instrument and is used in many genres, from classical to jazz and from jazz to pop. It is one of the most popular and recognisable instruments in the world. Despite this, I have noticed when using synthesizers to compose, that there are not many realistic

[1] Welsh, J. (2020). Caveman Flutists? First Instruments Date Back 40,000 Years. [online] livescience.com. Available at: https://www.livescience.com/20563-ancient-bone-flute.html [Accessed 11 Nov. 2019].

digital synthesizers for pianos. A synthesizer creates sounds from scratch based on digital and mathematical methods. The only method that seems to accurately recreate a piano, on a computer, is recordings of a piano. Some instruments can be easily depicted mathematically, but it appears that the piano is much harder to represent. There have been many attempts to do so and many are still ongoing, from companies such as Roland, but the piano remains a very complicated instrument to emulate. In this report, I will mathematically create a note and later compare it to a note recorded from a piano.

I will start by outlining the initial research that will be necessary to achieve my goal. This includes research on music theory, the piano, the physical and mathematical notions, that are used in the latter half of my study. Following this, I will focus on the algebraic development from simple formulae (Newton's second law and Hooke's law) to a more complex form (the wave equation), which can describe how the string will behave relative to time. The wave equation is too general to describe the movement of a piano string, so I will then specify the equation to fit the properties of a string. It will take into account, among other things, that the string is attached at both ends and that a piano string, in theory, produces an infinite number of different vibrations at once. The last step of my work will compare my mathematical representation to the acoustics of an actual piano. I will analyse the results and represent them visually to enable me to compare them. Finally, I will posit ideas on how to develop my equation further.

Chapter 1

Music Theory

Music theory encapsulates a huge amount of varied and complicated information. In its simplest form, it describes how music works and can sometimes be linked to physics. The aspects of music theory that are important in this report, are what sound is and how it works, as well as the harmonic series.

1.1 Vibration and Sound

Sound is a "mechanical disturbance from a state of equilibrium that propagates through an elastic material medium"[1], meaning that any vibration which can travel through a gas, a liquid or a solid is a sound. Humans can only hear some sounds. When the air vibrates enough and at a pitch or frequency that will move our eardrum, then our brain will interpret that movement as sound. The average human can hear anything above 0 dBs. A decibel is a very complex unit of measurement, that is used to measure many things, but here it designates the intensity of the sound. As for frequency (measured in hertz), the average human can also hear between 20Hz and 20KHz. It is interesting to note that frequency is measured as the reciprocal of time, its units being $[s^{-1}]$ (how many times something occurs in a second). This foreshadows that most of the mathematics in my report will involve time as a dimension.

[1] Encyclopedia Britannica. (2020). Sound — Properties, Types, and Facts. [online] Available at: https://www.britannica.com/science/sound-physics [Accessed 9 Dec. 2019].

1.2 Harmonic Series

The harmonic series is as mathematical as it is musical. Harmonic can refer to an oscillation or wave, and a series is a sum of terms from a sequence, which is a set of listed elements. Whenever a physical instrument makes a sound it produces an infinite number of different pitches (frequencies) that are all heard simultaneously, these pitches are all part of the harmonic series. This phenomenon is easiest to understand by imagining a string (the frequency at which the string oscillates translates to the pitch we can hear). In figure 1.1, all the different frequencies that are normally heard together can be observed individually.

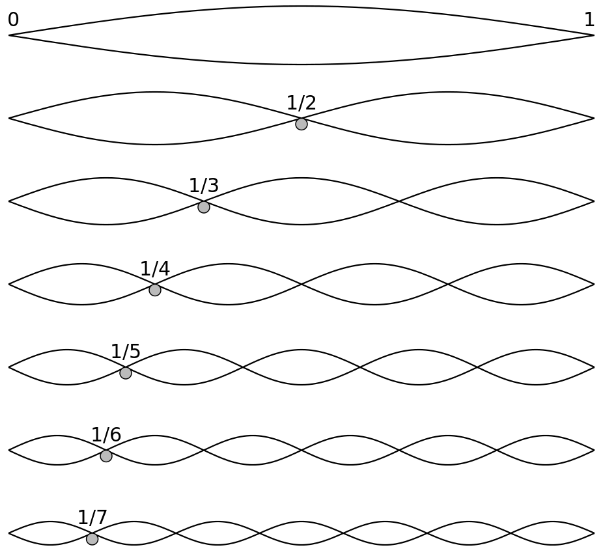

Figure 1.1: A visual aid for harmonics

If the note played is a middle A, which is tuned to 440Hz, its fundamental tone is 440Hz. It is represented by the first term of the sequence in figure 1.1, that would oscillate at this frequency. The 1st harmonic, right underneath it would oscillate with $\frac{1}{2}$ the wavelength and twice the frequency the second harmonic with a $\frac{1}{3}$ the wavelength and thrice the frequency, etc... The movement of the string is an infinite sum of these movements. Mathematically the harmonic series can be understood as the sum of each wavelength, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc... The harmonic series is:

$$S_n = \sum_{n=1}^{\infty} \frac{1}{n} \tag{1.1}$$

An important thing to understand is that an instrument will not play the fundamental tone and 1st harmonic at the same intensity, the intensity for each term of the harmonic series is different. The intensities are generally descending, but not always. How these intensities are distributed explains why different instruments sound they way the do, why a piano sounds like a piano and a guitar like a guitar.

Chapter 2

The Piano

I will be using a grand piano, as seen in figure 2.1 for recording samples, which will be especially important when I compare my mathematical version of a note to a real one. The main parts of the piano that will be considered are the hammer and the strings.

Figure 2.1: A grand piano

2.1 The Hammer

The workings of a piano hammer are surprisingly complex. To enable me to model a note, I will only take into consideration certain aspects of the mechanism seen in figure 2.2. The job of the hammer is to translate the motion of the piano key, pressed by the player into an impulse sent into the string. The hammer is what sets our string into motion, in other words, it determines how the string starts moving.

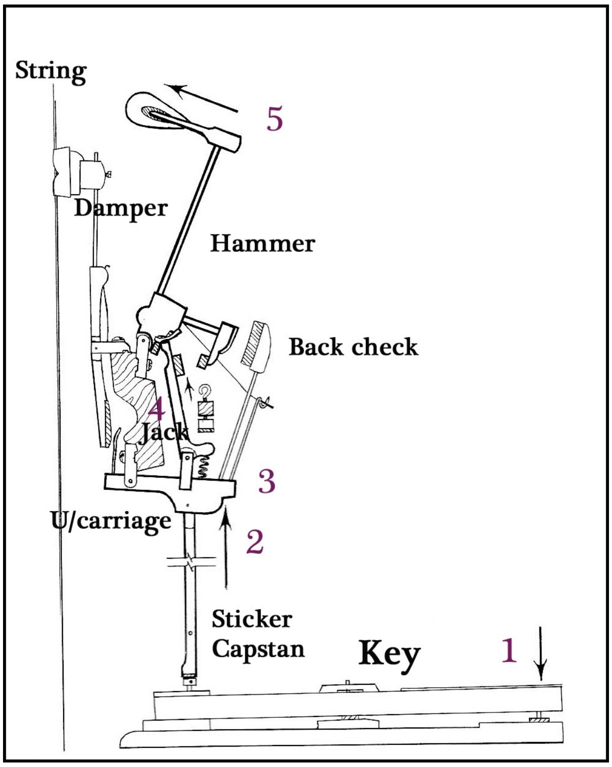

Figure 2.2: A piano hammer mechanism

When the key is pressed, the hammer hits the string and the damper lifts. The hammer hitting the string is what makes it vibrate. This vibration continues whilst the key is still pressed. Once the key is released, the damper returns to the initial position, stopping the string from moving and choking the sound, thus controlling the beginning and end of the note. The initial conditions of the string will later be determined by the behaviour of the hammer (cf. section 6.3).

2.2 The Strings

The strings are what initially creates the sound of the piano. The piano strings vary in linear density (thickness), length, rigidity, tension, and even in number. For the lowest notes, there is often one string, whereas for higher notes three strings are strung for one key. The number of strings per key is variable and depends on the piano. All these elements influence the pitch or timbre of the note. Strings are an amazing vessel through which vibration can pass. However, the vibrations are able to spread further, throughout the instrument, rather than just the strings. They propagate to the frame of the instrument and the soundboard which is specifically made to amplify the sound from the strings. The strings of a piano are always fixed at both ends. This determines our boundary conditions (cf. 6.2) and also that the string will sustain its oscillation as long as the key is held down. The movement of the string will not continue forever because there will inevitably be a loss of energy due to friction with the air. The pitch of the string, f, can be calculated with a simple equation and f can be found through simple measurements.

$$f = \frac{1}{2L}\sqrt{\frac{T}{\lambda}} \tag{2.1}$$

With L being the length of the string, T being its tension and λ being its linear density.

Chapter 3

Physical Notions

In this chapter, I will explain all the main physical notions and concepts that I will be using in chapter 5 and onwards.

3.1 Mechanical Waves

There are different kinds of waves. Electromagnetic and optic waves are both examples of waves. Generally speaking, waves are a "propagation of disturbances from place to place in a regular and organized way"[1] . The word propagation is important as it describes the movement of the wave. It is the movement of information through a medium (or vacuum in some cases) but the medium itself will finish where it started. For example, if you shake a string, the crest you create travels through the string (the medium) but the string itself finishes where it starts. Another example is sound, which is the changing pressure in the air. It is the difference in pressure that travels through the air, not the molecules themselves, as the air molecules all finish roughly where they started.

The waves that are important to this work are mechanical. Mechanical waves can only travel through matter. Neither electromagnetic or optic waves necessarily travel through matter. When people think of waves, they typically think of mechanical waves. They are, for the most part, easily observable or felt. A simple example is a piano string. When the hammer hits the string, every point along the string moves,

[1] Encyclopedia Britannica. (2020). wave — Behavior, Definition, and Types. Available at: https://www.britannica.com/science/wave-physics [Accessed 8 Jan. 2020].

propagating the movement. This creates a sound, which is another example that we cannot see, but can hear.

It is important to note that mechanical waves can be categorised into longitudinal and transverse waves, as illustrated in figure 3.1. In a transverse waves, the points of the wave oscillate perpendicular to the propagation of the information and in a longitudinal wave it vibrates in the direction of propagation. In many circumstances, these two kinds of waves coexist. Sound is an example of a longitudinal wave and the movement in the shaken string, that I described earlier, is a transverse wave. Transverse waves can occur in two dimensions.

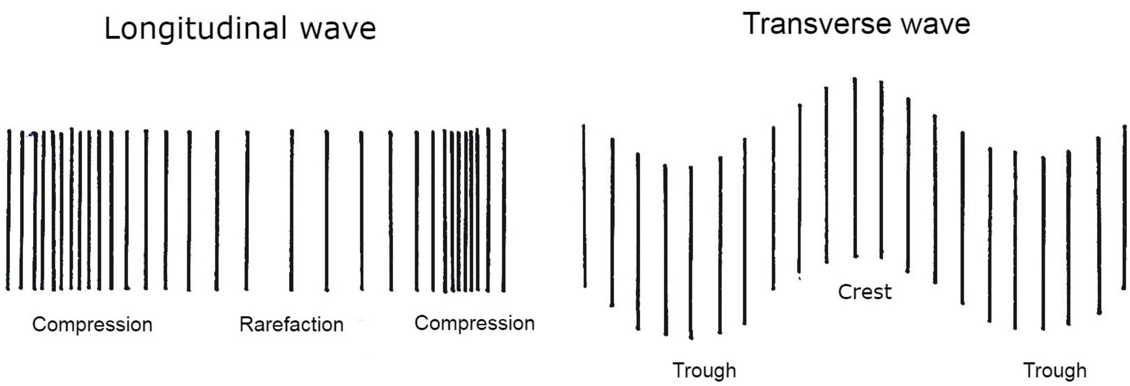

Figure 3.1: Longitudinal and transverse waves

3.2 Newton's Second Law

Newton's second law is :

$$F = ma \tag{3.1}$$

Force, F, is equal to mass, m, multiplied by acceleration, a, and is the basis of a huge part of classical physics. It is a fundamental relationship that can be modified to apply to many different situations. Its simplicity belies its varied applications.

3.3 Hooke's Law

Hooke's law is:

$$F = kd \tag{3.2}$$

Hooke's law applies to potential kinetic force in springs. Force is equal to displaced distance multiplied by k. k is the stiffness of the spring, it is measured in $[N/m]$ (newtons per meter). It is a constant, specific to each spring. It is thanks to a combination of this formula and newton's second law that we find equation 2.1 and also what we use to obtain the wave equation in chapter 5.

Chapter 4

Mathematical Notions

In this chapter, I will explain all the main mathematical notions and concepts that I will be using in chapter 5 and onwards.

4.1 Sine Waves

Sine waves are a very useful kind mathematical function, which describe many periodic phenomena (I will not be explaining cosine waves, as they are similar to sine waves in all respects apart from being a horizontal translation of a sine wave). Waves are all based on sinusoidal functions. These functions oscillate and are continuous. Sine functions are ubiquitous in maths and physics because periodic movement can be found in nature, both on a microscopic and a macroscopic level. The general form of a sine function is:

$$f(x) = Asin(\omega x + \varphi) + \phi \tag{4.1}$$

A, ω, φ and ϕ are all constants that change the sine wave's behaviour. Only A and ω will be needed in this paper. A is the amplitude, changing the height of the minima and maxima of the function and ω is the angular frequency and it changes the number of oscillations in a particular interval. Sine Waves are essential to Fourier, which will be further explained in section 4.4

4.2 Partial Derivatives

For any function $f(x)$ the derivative is $\frac{df}{dx}$. A derivative is "the rate of change of the function with respect to the independent variable"[1]. Not all functions have a single variable though and in section 4.3 we introduce the wave equation which has a function, $\mu(x,t)$, that has two variables. We cannot differentiate it in the same way as $f(x)$. To determine the derivative of a function such as μ we need two tangents, one for each variable. The method used to differentiate a function with two or more variables, with respect to one of these variables, is called partial derivation. It is written similarly to a normal derivative but the d is written as ∂. In addition, the derivation is done in multiple parts, one part for each variable, which for our example gives: $\frac{\partial\mu}{\partial x}$ and $\frac{\partial\mu}{\partial t}$. To calculate $\frac{\partial\mu}{\partial x}$ for example, you treat t as a constant and derive the rest of the function in terms of x and then vice-versa. As for second-order partial derivatives there are four possibilities. Deriving by a variable twice as for $\frac{\partial^2\mu}{\partial x^2}$ and $\frac{\partial^2\mu}{\partial y^2}$, or deriving by one variable and the another as for $\frac{\partial^2\mu}{\partial x\partial y}$ and $\frac{\partial^2\mu}{\partial y\partial x}$, which are called mixed partial derivatives.

4.3 Wave Equations

The wave equation, or classical wave equation, takes the form of:

$$\frac{\partial^2\mu}{\partial t^2} = c^2\frac{\partial^2\mu}{\partial x^2} \tag{4.2}$$

With μ being a function of time and space. This equation is developed in chapter 5 from the equations in section 3.2 and 3.3. The function $\mu(x,t)$ possesses two variables; x that represents distance and t that represents time. Figure 4.1 is a visual aid to understand how it works. It is important to understand that that the t axis represents the passage of time and that the output is given by y.

A very important property of the wave equation is that it is linear, which means the superposition principle is applicable. "When two waves interfere, the resulting displacement of the medium at any location is the algebraic sum of the displacements of the individual waves at that same location"[2]. This property is significant in chapter 6.

[1]Borowski E.J. and Borwein J.M., Dictionnary of Mathematics 1st ed. Harper Collins, 1989.

[2]Physicsclassroom.com. 2020. Physics Tutorial: Interference Of Waves. [online] Available at: https://www.physicsclassroom.com/class/waves/Lesson-3/Interference-of-Waves [Accessed 6 January 2020].

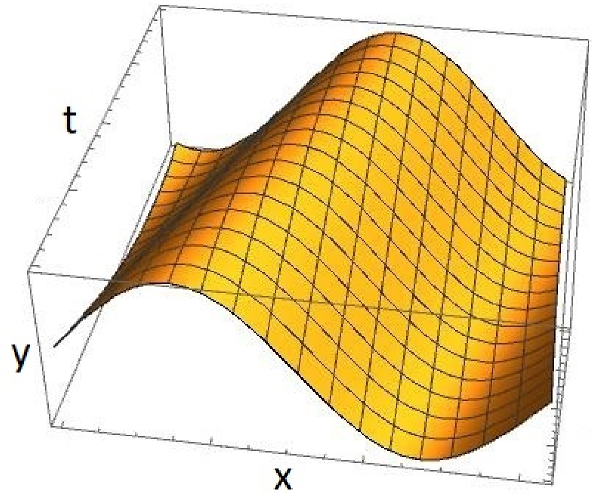

Figure 4.1: A visual aid to understand the wave equation

4.4 Fourier

The Fourier theorem is "a mathematical theorem stating that a periodic function $j(x)$ which is reasonably continuous may be expressed as the sum of a series of sine or cosine terms"[3]. The Fourier series is an infinite sum of sine and cosine functions as seen below:

$$j(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cos(\frac{nx\pi}{L}) + b_n sin(\frac{nx\pi}{L})) \tag{4.3}$$

This is the general form of the Fourier Series. $2L$ is the period of the function j. The harmonic series (cf. 1.2) gives us the wavelengths for each value of n. a_n and b_n are the Fourier coefficients, which are specific to each situation and are what makes a Fourier series resemble a periodic and reasonably continuous graph. They are given by the equations:

[3] Sfu.ca. (n.d.). Fourier-Theorem. [online] Available at: https://www.sfu.ca/sonic-studio-webdav/handbook/Fourier_Theorem.html [Accessed 10 Feb. 2020].

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx \\
a_n &= \frac{2}{L}\int_0^L f(x)cos(\frac{nx\pi}{L})dx \\
b_n &= \frac{2}{L}\int_0^L f(x)sin(\frac{nx\pi}{L})dx
\end{aligned}
\tag{4.4}
$$

Using the Fourier Series you can build up functions. This is called Fourier synthesis, which will be used in chapter 6. The opposite is called the Fourier transform, which is taking a complex function or waveform and breaking it down into sine and cosine functions. We will be doing a simplified version of this by plotting a spectrum in chapter 7.

4.5 Complex Numbers

The square root of a negative number has no real solution. Historically, this has been problematic. To resolve this issue, complex numbers were invented. A complex number has a real and imaginary component. The imaginary number is stated as $i = \sqrt{-1}$. Complex numbers are often represented on the complex plane, where one axis represents real numbers and the other, imaginary numbers (cf. fig 4.2).

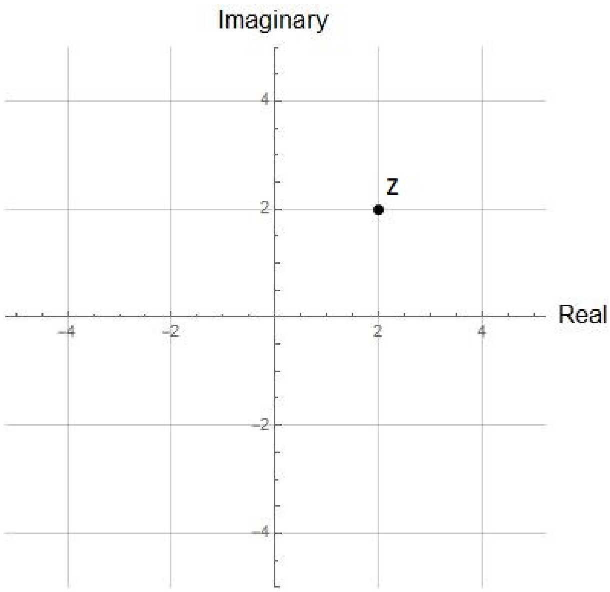

Figure 4.2: The coordinate z in the complex plane

Complex numbers can be written in three different forms: the general form, the polar form and the exponential form. The coordinate z in figure 4.2 in these three forms is:

$$\begin{aligned} z &= 2+2i \\ z &= 2\sqrt{2}\big(Cos(\frac{\pi}{4})+iSin(\frac{\pi}{4})\big) \\ z &= 2\sqrt{2}e^{i\frac{\pi}{4}} \end{aligned} \tag{4.5}$$

The relationship between the polar form and the exponential form is:

$$q\big(Cos(\eta)+iSin(\eta)\big) = qe^{i\eta} \tag{4.6}$$

Chapter 5

From Newton to Alembert

This section explains how simple laws such as Newton's second law of motion, $F = ma$, and Hooke's law, $F = kd$, can be used to develop Jean le Rond d'Alembert's version of the wave equation, $\frac{\partial^2\mu}{\partial t^2} = c^2\frac{\partial^2\mu}{\partial x^2}$. Below is a diagram showing a string conceptualised as a certain number of particles connected by springs.

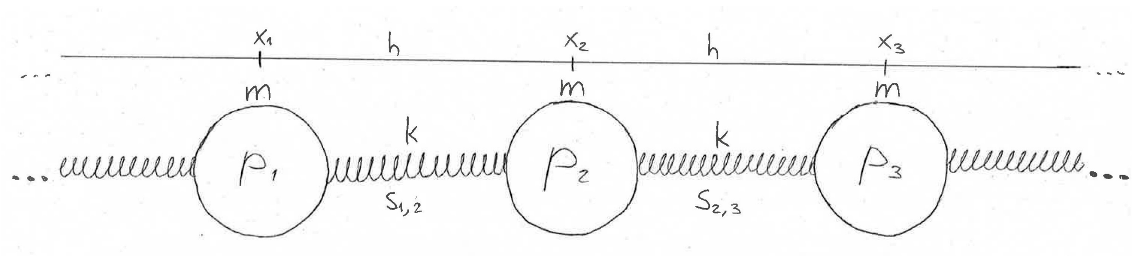

Figure 5.1: A diagram of a conceptualised string

In the figure 5.1 the conceptualised string, with particles, $p_1, p_2...$ of mass m, at a distance h from one another are held together by springs $s_{1,2}, s_{2,3}...$ of rigidity k. The particles are initially positioned at coordinates x_1, x_2, ... and move vertically. This means that the disturbance travelling through the string will do so as a transverse wave. If p_2, for example, is pulled upwards, the springs $s_{1,2}$ and $s_{2,3}$ will want to pull it towards the particles p_1, p_2 so the horizontal forces will cancel out. Vertically, however, it will accelerate downwards. It's movement will propagate and all the particles will be set into motion. The acceleration can be described by Newton's second law. The force exerted by the springs can be described by Hooke's law. With this information, it is possible to develop the wave equation.

Newton's second law, which has previously been elaborated on in section 3.2, initially needs to be developed into a function in terms of distance (x) and time (t). Given that the acceleration acting on particle p_2 is the second derivative of distance with respect to time, we transform it into a function of distance with respect to time for the particle p_2 ($\mu_{x2}(t)$).

$$F = ma(t) = m\frac{d^2\mu_{x2}}{dt^2}(t) \tag{5.1}$$

Our function μ still needs to be a function of space and time, not only of time. Given the function is specific to the movement of the particle p_2, we add x_2 to the function, not as a variable but as a coordinate (so we are not actually changing our function). This still turns our derivative into a partial derivative as we now of two elements in our function.

$$F = m\frac{d^2\mu_{x2}}{dt^2}(t) = m\frac{\partial^2}{\partial t^2}\mu(x_2, t) \tag{5.2}$$

Next, we need to tackle the forces described by Hooke's law, which has previously been elaborated on in section 3.3. This is needed to describe the forces of the springs acting on the particle p_2. There is a spring on either side of it, $s_{1,2}$ and $s_{2,3}$. The displacement of the springs are $d_{1,2}$ and $d_{2,3}$, which is how much the spring is stretched beyond it's normal length. Below, Hooke's law will be applied to the srpings on both sides.

$$\vec{F} = \begin{bmatrix} \vec{F_x} \\ F_y \end{bmatrix} + \begin{bmatrix} -\vec{F_x} \\ F_y \end{bmatrix} = 2\vec{F_y} = k\vec{d_{1,2_y}} + k\vec{d_{2,3_y}} \tag{5.3}$$

The vertical displacements $d_{1,2_y}$ and $d_{2,3_y}$ are the difference in height between to particles. This can be calculated by difference of μ for two particles.

$$k\vec{d_{1,2_y}} + k\vec{d_{2,3_y}} = k(\mu(x_3, t) - \mu(x_2, t)) - k(\mu(x_2, t) - \mu(x_1, t)) \tag{5.4}$$

In equation 5.2 we have the force as described by newton's second law and in 5.4 we have the forces of the springs as described by Hooke's law. Given that they are the only forces acting in this system and there are no losses of force we can equate them.

$$m\frac{\partial^2}{\partial t^2}\mu(x_2,t) = k(\mu(x_3,t) - \mu(x_2,t)) - k(\mu(x_2,t) - \mu(x_1,t)) \tag{5.5}$$

Rearranging the equation:

$$\frac{\partial^2}{\partial t^2}\mu(x_2,t) = \frac{k}{m}(\mu(x_3,t) - 2\mu(x_2,t) + \mu(x_1,t)) \tag{5.6}$$

Next, K, M and L are introduced. Respectively they designate the total rigidity of string, its total mass and its total length; they are all fixed values. N designates the number of particles in the string. In 5.7, $K = \frac{k}{N}$[1] is less intuitive than the other two formulae. This relationship is due to how the rigidity in springs behave in a series (connected). They follow the law $\frac{1}{K} = \frac{1}{k_1} + \frac{1}{k_2} + ...$

$$K = \frac{k}{N} \quad \text{and} \quad M = Nm \quad \text{and} \quad L = Nh \tag{5.7}$$

By using the formulae above we can find $\frac{k}{m}$ in terms of K, L, M and h.

$$\frac{k}{m} = \frac{KN}{\frac{M}{N}} = \frac{KN^2}{M} = \frac{KL^2}{Mh^2} \tag{5.8}$$

5.8 is substituted into 5.6:

$$\frac{\partial^2}{\partial t^2}\mu(x_2,t) = \frac{KL^2}{M}\frac{\mu(x_3,t) - 2\mu(x_2,t) + \mu(x_1,t)}{h^2} \tag{5.9}$$

Next, given that x_1 is treated as the origin and the other coordinates are spaced away from it by h, $2h$, etc... , we can replace $x_1, x_2, x_3, ...$ by $x, x+h, x+2h, ...$ To make the conceptualised string more analogous to an actual string, we make the number of particles, N, approach ∞. The length L of the string is constant so as $N \to \infty$, $h \to 0$ ($L = Nh$). Below the limits have been added to both sides of the equation. The following lines show the simplification via differentiation. Once this has been done x will finally be a variable in the function μ and no longer coordinates.

[1] $\frac{1}{K} = N\frac{1}{k}$

$$\lim_{h\to 0}\frac{\partial^2}{\partial t^2}\mu(x+h,t)=\frac{KL^2}{M}\lim_{h\to 0}\frac{\mu(x+2h,t)-2\mu(x+h,t)+\mu(x,t)}{h^2} \tag{5.10}$$

$$\frac{\partial^2}{\partial t^2}\mu(x,t)=\frac{KL^2}{M}\lim_{h\to 0}\frac{\frac{\mu(x+2h,t)-\mu(x+h,t)}{h}-\frac{\mu(x+h,t)-\mu(x,t)}{h}}{h}$$

$$\frac{\partial^2}{\partial t^2}\mu(x,t)=\frac{KL^2}{M}\lim_{h\to 0}\frac{\frac{\partial}{\partial x}\mu(x+h,t)-\frac{\partial}{\partial x}\mu(x,t)}{h}$$

$$\frac{\partial^2}{\partial t^2}\mu(x,t)=\frac{KL^2}{M}\frac{\partial^2}{\partial x^2}\mu(x,t)$$

The last step to obtain Jean le Rond d'Alembert's version of the wave equation is to consider $\frac{KL^2}{M}$. By using the units of K, L and M $[N/m, m, kg]$, I will demonstrate below how $\frac{KL^2}{M}$ and c^2 are equivalent. c $[m/s]$ represents the speed of propagation throughout the wave. This can be verified by different uses of the wave equation.

$$\frac{KL^2}{M}=\frac{[N/m][m]^2}{[kg]}=\frac{[N][m]}{[kg]}=\frac{\frac{[kg][m]}{[s^2]}[m]}{[kg]}=\frac{[kg][m^2]}{[kg][s^2]}=[m/s]^2=c^2 \tag{5.11}$$

5.10 is substituted into 5.11 :

$$\frac{\partial^2\mu}{\partial t^2}=c^2\frac{\partial^2\mu}{\partial x^2} \tag{5.12}$$

Chapter 6

From The Wave Equation To The String

In this chapter, the wave equation is adapted to cater to strings. We will first look for all the solutions to the wave equation in section 6.1. Then, in sections 6.2 and 6.3, we will reduce the number of solutions by applying what we know about our string. This will then be possible to solve in section 6.4

6.1 Separating The Variables

We start by searching for all the solutions to the wave equation. The first thing we need to do is separate the variables, as it is difficult to work with multiple variable functions. Separating the variables would have limited the solutions to the equation had we not been able to separate all x and t variables to different sides of the equation in 6.2. We introduce two new functions:

$$\mu(x,t) = a(t)b(x) \tag{6.1}$$

The wave equation (cf. 5.12) can be rewritten with these new functions. They are moved to the left or right hand side of the equation, according to their variables. On the left there will only be the functions of time and on the right of distance.

$$\frac{1}{c^2}\frac{a''(t)}{a(t)} = \frac{b''(x)}{b(x)} \tag{6.2}$$

Given that each side is only dependant on one variable and is equal to the other side, which is not dependant on that variable, we know that both sides are constant. This constant is defined as ω, which could be equal to anything (it could even be a complex number). By rearranging our equations two second order linear differential equations are found.

$$\frac{1}{c^2}\frac{a''(t)}{a(t)} = \omega \qquad \frac{b''(x)}{b(x)} = \omega \tag{6.3}$$

$$a''(t) - \omega c^2 a(t) = 0 \qquad b''(x) - \omega b(x) = 0$$

To solve our second order linear differential equation we need to replace each function by values. As the derivative of the exponential function e^x is itself, it is a perfect candidate. We try $a(t) = e^{pt}$ and $b(x) = e^{qx}$. We have introduced p and q, below we will find their solutions so that later they can be eliminated.

$$(e^{pt})'' - \omega c^2 e^{pt} = 0 \qquad (e^{qx})'' - \omega e^{qx} = 0 \tag{6.4}$$

$$p^2 e^{pt} - \omega c^2 e^{pt} = 0 \qquad q^2 e^{qx} - \omega e^{qx} = 0$$

$$p^2 = \omega c^2 \qquad q^2 = \omega$$

$$p = \pm c\sqrt{\omega} \qquad q = \pm\sqrt{\omega}$$

Given the solutions from 6.4, we have to replace with both negative and positive possibilities. To generalise our solutions, the constants v_1 to v_4 are added, which are arbitrary.

$$a(t) = v_1 e^{c\sqrt{\omega}t} + v_2 e^{-c\sqrt{\omega}t} \qquad b(x) = v_3 e^{\sqrt{\omega}x} + v_4 e^{-\sqrt{\omega}x} \qquad \forall \qquad \omega \neq 0 \tag{6.5}$$

$\omega \neq 0$ because it would give us a function of the wave equation with no movement. For $\omega = 0$, a different way is needed to find a solution. Below, given $\omega = 0$, the terms including ω are eliminated and we are left with the second derivative of each function which equals 0. The general solution (the final line) is the general form of the second integral of 0 with the same arbitrary v_1 to v_4:

$$a''(t) - \omega c^2 a(t) = 0 \qquad b''(x) - \omega b(x) = 0 \tag{6.6}$$

$$a''(t) = 0 \qquad b''(x) = 0$$

$$a(t) = v_1 t + v_2 \qquad b(x) = v_3 x + v_4$$

Below, we have all the solutions to the wave equation:

$$\begin{aligned} &\mu(x,t) = (v_1 e^{c\sqrt{\omega}t} + v_2 e^{-c\sqrt{\omega}t})(v_3 e^{\sqrt{\omega}x} + v_4 e^{-\sqrt{\omega}x}) \quad \forall \quad \omega \neq 0 \\ &\mu(x,t) = (v_1 t + v_2)(v_3 x + v_4) \quad \text{for} \quad \omega = 0 \end{aligned} \tag{6.7}$$

6.2 Boundary Conditions

The solutions we have found are not specific to a piano string, we know a piano string is fixed in two places (cf. section 2.2 and figure 6.1), we must then specify our solutions to cater to these boundary conditions.

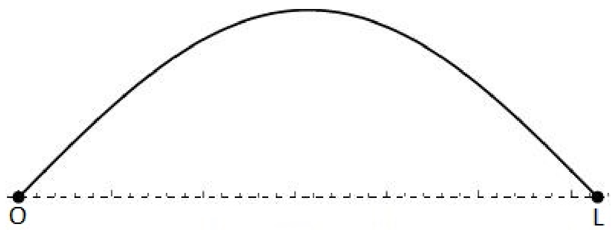

Figure 6.1: A diagram of a string

The boundary conditions are expressed mathematically below:

$$\mu(0,t) = 0 \qquad \mu(L,t) = 0 \qquad \forall \quad t \geq 0 \tag{6.8}$$

Given what was established in equation 6.1:

$$b(0) = 0 \qquad b(L) = 0 \tag{6.9}$$

For the specific case where $\omega = 0$, the general form of $b(x)$, found in part 6.7, is an affine function. The only function that touches the x axis at $x = 0$ and $x = L$ is $b(x) = 0$. There is no displacement in this situation so we ignore it.

For the case where $\omega \neq 0$, the situation satisfying $b(0) = 0$ is found when:

$$b(0) = v_3 e^{\sqrt{\omega}0} + v_4 e^{-\sqrt{\omega}0} = 0 \tag{6.10}$$

$$v_3 e^0 + v_4 e^0 = 0$$

$$v_3 = -v_4$$

Next, for $b(L) = 0$ to be satisfied:

$$b(L) = v_3 e^{\sqrt{\omega}L} + v_4 e^{-\sqrt{\omega}L} = 0 \tag{6.11}$$

We use the property found in 6.10

$$v_3(e^{\sqrt{\omega}L} - e^{-\sqrt{\omega}L}) = 0 \tag{6.12}$$

$$e^{\sqrt{\omega}L} - e^{-\sqrt{\omega}L} = 0$$

$$e^{\sqrt{\omega}L} = e^{-\sqrt{\omega}L}$$

Each side is multiplied by $e^{\sqrt{\omega}L}$

$$e^{2\sqrt{\omega}L} = 1 \tag{6.13}$$

Using complex numbers, we can define $\sqrt{\omega}$, giving us an infinite number of solutions. First,the right hand side is turned into its complex form and then $\sqrt{\omega}$ is isolated.

$$e^{2\sqrt{\omega}L} = e^{2k\pi i} \qquad \text{such that} \qquad k \in \mathbb{Z} \tag{6.14}$$

$$2\sqrt{\omega}L = 2k\pi i$$

$$\sqrt{\omega} = \frac{k\pi i}{L}$$

We then substitute our solution from 6.14 into the general solution from the separation of the variables (cf. 6.7), keeping k as defined and without taking into account $\omega = 0$, which has already been disregarded.

$$\mu(x,t) = v_3(e^{\frac{k\pi i}{L}x} - e^{-\frac{k\pi i}{L}x})(v_1 e^{\frac{k\pi ic}{L}t} + v_2 e^{-\frac{k\pi ic}{L}t}) \tag{6.15}$$

We develop our equation using equation 4.6:

$$\mu(x,t) = v_3\Big(cos(\frac{k\pi}{L}x) + isin(\frac{k\pi}{L}x) - cos(-\frac{k\pi}{L}x) - isin(-\frac{k\pi}{L}x)\Big)$$
$$\bigg(v_1\Big(cos(\frac{k\pi c}{L}t) + isin(\frac{k\pi c}{L}t)\Big) + v_2\Big((cos(-\frac{k\pi c}{L}t) + isin(-\frac{k\pi c}{L}t)\Big)\bigg) \quad (6.16)$$

It is then rearranged and simplified:

$$\mu(x,t) = 2iv_3sin(\frac{k\pi}{L}x)\bigg(v_1cos(\frac{k\pi c}{L}t) + v_2cos(-\frac{k\pi c}{L}t)$$
$$+ v_1isin(\frac{k\pi c}{L}t) + v_2isin(-\frac{k\pi c}{L}t)\bigg) \quad (6.17)$$

$$\mu(x,t) = 2iv_3sin(\frac{k\pi}{L}x)\bigg((v_1+v_2)cos(\frac{k\pi c}{L}t) + (v_1-v_2)isin(\frac{k\pi c}{L}t)\bigg)$$

We simplify it further, a_k and b_k are defined as below:

$$\begin{gathered}\mu(x,t) = sin(\frac{k\pi}{L}x)\bigg(a_kcos(\frac{k\pi c}{L}t) + b_ksin(\frac{k\pi c}{L}t)\bigg) \\ a_k = 2iv_3(v_1+v_2) \qquad b_k = -2v_3(v_1-v_2)\end{gathered} \quad (6.18)$$

All our solutions for $\mu(0,t)$ and $\mu(L,t)$ equal 0, for all values of k, so no matter how we sum them, the boundary conditions will still be met. Our equation is generalised by making it an infinite sum. This we can do because the wave equation is linear and thus obeys the superposition principal (c.f. 4.3) This equation now takes into account all solutions that obey the boundary conditions.

$$\mu(x,t) = \sum_{k=1}^{\infty} sin(\frac{k\pi}{L}x)\bigg(a_kcos(\frac{k\pi c}{L}t) + b_ksin(\frac{k\pi c}{L}t)\bigg) \quad (6.19)$$

6.3 Initial Conditions

The final aspects to consider are the initial conditions, which will give us more information on a_k and b_k. The initial conditions are the shape of the string and its speed at $t = 0$. The solution to $\mu(x, 0)$ is the shape of the string at $t = 0$ which we will name $g(x)$. The function μ'_t is the function μ derived uniquely by the variable of time, thus being speed in function of x and t. When $t = 0$ this function is named $h(x)$. These functions will be defined in section 6.4.

$$\mu(x,0) = g(x) \qquad \mu'_t(x,0) = h(x) \tag{6.20}$$

We substitute $t = 0$ into 6.19:

$$\mu(x,0) = \sum_{k=1}^{\infty} sin(\frac{k\pi}{L}x)\Big(a_k cos(\frac{k\pi c}{L}0) + b_k sin(\frac{k\pi c}{L}0)\Big) \tag{6.21}$$

$$\mu(x,0) = \sum_{k=1}^{\infty} a_k sin(\frac{k\pi}{L}x) = g(x)$$

And substitute $t = 0$ into the derivative (uniquely by time) of 6.19:

$$\mu'_t(x,0) = \sum_{k=1}^{\infty} sin(\frac{k\pi}{L}x)\Big(-a_k sin(\frac{k\pi c}{L}0)\frac{k\pi c}{L} + b_k cos(\frac{k\pi c}{L}0)\frac{k\pi c}{L}\Big) \tag{6.22}$$

$$\mu'_t(x,0) = \sum_{k=1}^{\infty} b_k sin(\frac{k\pi}{L}x)\frac{k\pi c}{L} = h(x)$$

Given that what we ended up with is so similar to Fourier as seen in 4.3 and 4.4, we can use the same process to find a_k and b_k:

$$a_k = \frac{2}{L}\int_0^L g(x)sin(\frac{k\pi x}{L})dx \quad b_k = \frac{2}{ck\pi}\int_0^L h(x)sin(\frac{k\pi x}{L})dx \tag{6.23}$$

Our solution is:

$$\begin{gathered} \mu(x,t) = \sum_{k=1}^{\infty} sin(\frac{k\pi}{L}x)\left(a_k cos(\frac{k\pi c}{L}t) + b_k sin(\frac{k\pi c}{L}t)\right) \\ a_k = \frac{2}{L}\int_0^L g(x)sin(\frac{k\pi x}{L})dx \qquad b_k = \frac{2}{ck\pi}\int_0^L h(x)sin(\frac{k\pi x}{L})dx \end{gathered} \tag{6.24}$$

6.4 Solution For A String

To apply this equation to a piano string we need to carefully select the starting conditions. In a piano, the string is initially stationary, this means $g(x) = 0$. The hammer hits the string setting it into motion. More specifically, "the hammer strike point for our standard piano (and for most pianos) is at about the one-eight point"[1]. The velocity, however, depends on how hard the key is played. We will assume that the hammer hits the string at 3m/s, as this is an average speed[2]. We set $L = 1[m]$ (the length of the string) and $2\delta = 0.01[m]$(the width of the hammer).

The boundary conditions are:

$$\mu(0,t) = 0 \qquad \mu(L,t) = 0 \tag{6.25}$$

The initial conditions are:

$$\mu(x,0) = g(x) = 0 \qquad \mu'_t(x,0) = h(x) = \begin{cases} 0 & \text{for } [0;L] \setminus [\frac{1}{8}+\delta; \frac{1}{8}-\delta] \\ 3 & \text{for } [\frac{1}{8}+\delta; \frac{1}{8}-\delta] \end{cases} \tag{6.26}$$

[1]Giordano, Nicholas. (n.d.). Physics of the piano.
[2]Acs.psu.edu. (2020). [online] Available at: https://www.acs.psu.edu/drussell/Publications/pianohammer.pdf[Accessed 1 Mar. 2020].

Solving for a_k with our new values we get:

$$a_k = 2\int_0^1 g(x)sin(k\pi x)dx \tag{6.27}$$

$$a_k = 2\int_0^1 0sin(k\pi x)dx$$

$$a_k = 0$$

Solving for b_k with our new values we get:

$$b_k = \frac{2}{ck\pi}\int_0^1 h(x)sin(\frac{k\pi x}{1})dx \tag{6.28}$$

The separation of integrals and it's limits, refer to the piecewise function (cf. 6.28):

$$b_k = \frac{2}{ck\pi}\left(\int_0^{0.12} 0sin(k\pi x)dx + \int_{0.12}^{0.13} 3sin(k\pi x)dx + \int_{0.13}^{1} 0sin(k\pi x)dx\right) \tag{6.29}$$

$$b_k = \frac{2}{ck\pi}\int_{0.12}^{0.13} 3sin(k\pi x)dx$$

$$b_k = \frac{6}{ck\pi}\left[-\frac{cos(k\pi x)}{k\pi}\right]_{0.12}^{0.13}$$

$$b_k = \frac{6}{ck\pi}\left(\frac{-cos(\frac{k\pi 13}{100})}{k\pi} - \frac{-cos(\frac{k\pi 3}{25})}{k\pi}\right)$$

$$b_k = \frac{6}{ck\pi}\left(\frac{cos(\frac{k\pi 3}{25}) - cos(\frac{k\pi 13}{100})}{k\pi}\right)$$

Our two Fourier coefficients (a_k and b_k) are substituted into 6.24:

$$\mu(x,t) = \sum_{k=1}^{\infty}\frac{6}{ck\pi}\left(\frac{cos(\frac{k\pi 3}{25}) - cos(\frac{k\pi 13}{100})}{k\pi}\right)sin(k\pi x)sin(k\pi ct) \tag{6.30}$$

This is our final version of the wave equation. It takes into account the boundary and initial conditions and we can use it to predict how a string will move and the sound it will make.

Chapter 7

How The Solution Compares to Reality

In this chapter, we are going to compare our solution of the wave equation to a recording of a piano. I recorded the A4 note on a piano, that has a frequency of 440hz because it is the note to which most instruments are tuned (note that we are still using the values set in section 6.4). With this information we can also find the value for c:

$$f = \frac{Nc}{2L} \tag{7.1}$$

N is equal to the number of the term in the harmonic series and as we are looking for the fundamental tone $N = 1$ and f is the frequency. Rearranged we get:

$$c = \frac{f2L}{N} = \frac{440 \times 2 \times 1}{1} = 880m/s \tag{7.2}$$

The functions we finish with in equation 6.30 can be broken down into two parts:

$$sin(k\pi x)sin(k\pi ct) \tag{7.3}$$

And:

$$\frac{6}{ck\pi}\left(\frac{cos(\frac{k\pi 3}{25}) - cos(\frac{k\pi 13}{100})}{k\pi}\right) \tag{7.4}$$

Equation 7.3 is made up of two sinusoidal functions and in the infinite sum of 6.30, k tends towards infinity. This means that the wavelength changes for every value of k. The wavelength follows the harmonic series as seen in chapter 1.2. As for equation 7.4, it modulates the amplitudes for the different wavelengths.

To compare, our theoretical solution to reality, we first graph the amplitude relative to that of the fundamental tone for our theoretical solution in figure 7.1. All the values were multiplied by the inverse of the solution for $k = 1$ so that the fundamental tone would be our standard amplitude of one. Secondly, we plot the spectrum of our recording in figure 7.2.

In figure 7.1, it is interesting to note that at the 8th and 16th harmonics, the relative amplitude equals zero. This is to be expected given we struck the string at an 8th of its length.

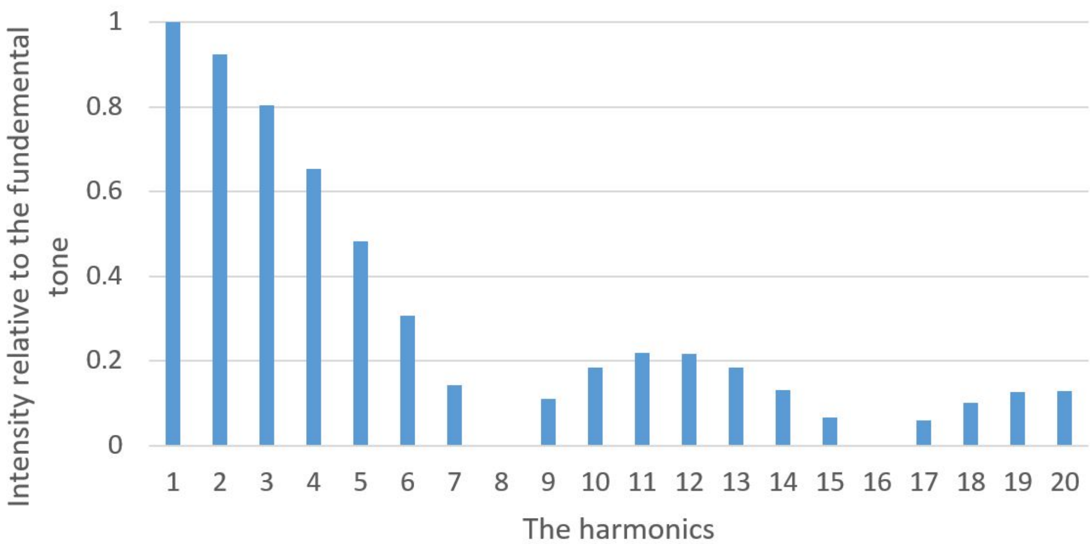

Figure 7.1: The relative amplitudes for each member of the harmonic sequence

Using Audacity, our recording of a note is analysed, with the *plot spectrum* tool. Figure 7.2, is the spectrum plotted. It gives us the intensity of the sound against the frequency. This is similar to the other graph, however, all the frequencies can be seen and not just the harmonic sequence. The highest peaks that are visible on the graph are the notes from the harmonic sequence. It is important to note that from 3500Hz onwards there is no information, which is probably due to the quality of the microphone used. This limits the number of harmonics (large peaks) we can see to seven (and it is possible that the 8th is equal zero, so we do not see it). As the axes on the figure are too small to be discernible, as Audacity has extremely limited aesthetic options. A larger version of the figure is available in the annexe A.

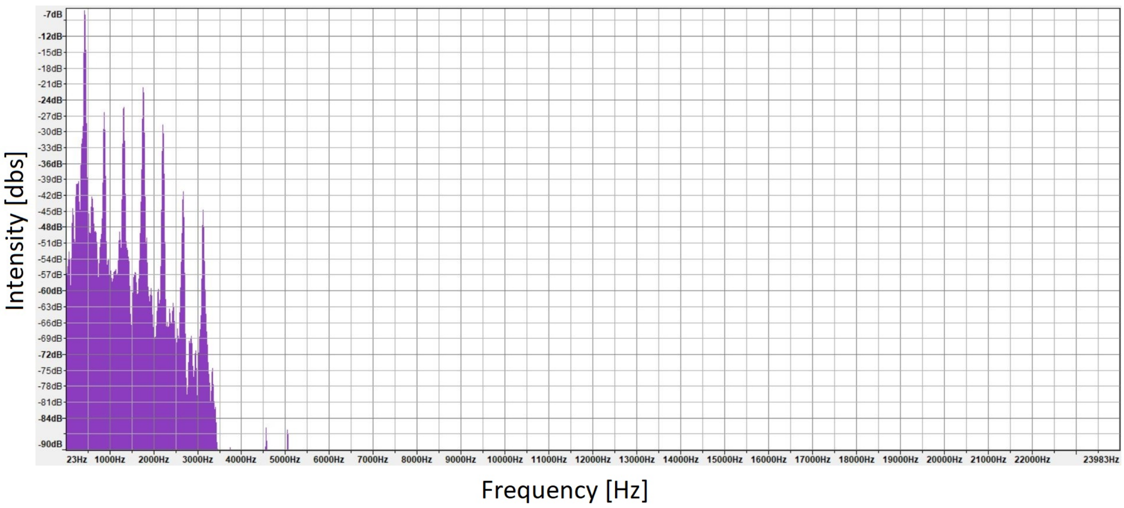

Figure 7.2: The spectrum of the recording

By using the tools in Audacity we can determine the frequency and intensity of the peaks in figure 7.2. Initially, the decibels were turned into a linear scale. All the values were then multiplied by the inverse of the first peak's intensity, in order to obtain amplitudes relative to that of the fundamental tone, which is one. This is what we did for figure 7.1. In figure 7.3 both sets of data are graphed simultaneously. Both the theoretical and acoustic values are mapped to the same linear scale so the comparison is valid.

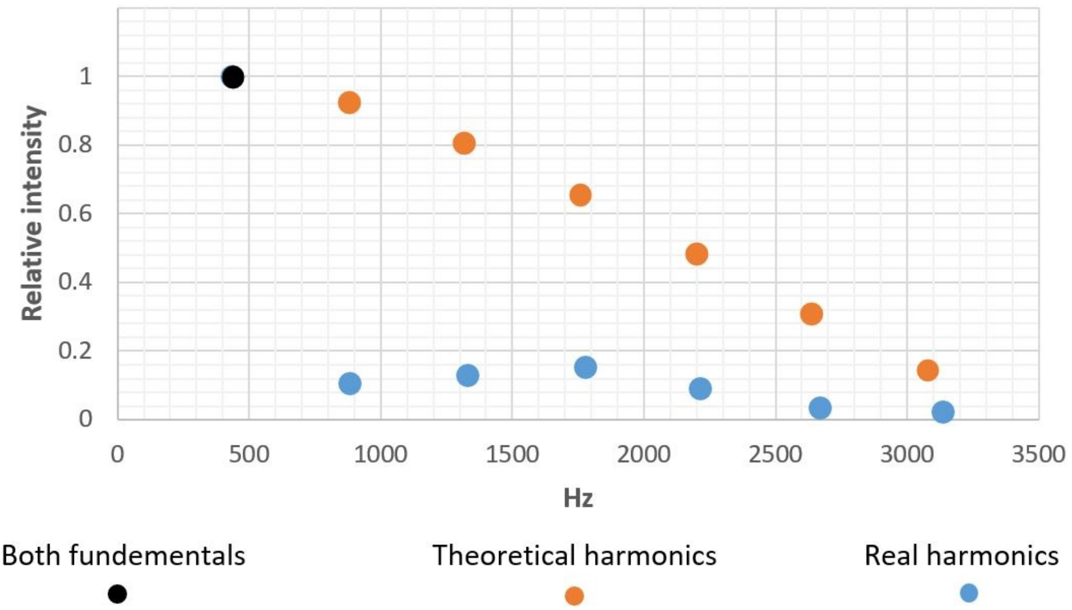

Figure 7.3: The theoretical and acoustic data graphed simultaneously

As can be seen in the diagram, our values differ considerably.

Conclusion

On one hand, the report was partly successful. Theoretically speaking, my answer is very promising. The fact that the 8th and 16th harmonics intensities are equal to zero is encouraging and also having animated the movement of the string on Mathematica, the movement seems plausible.

On the other hand, when it came to the comparison, the theoretical solutions were very far from reality. There is a certain consensus, that the piano is one of the hardest instruments to emulate. What surprised me in the comparison, however, was how quickly the theoretical and acoustic models of the string differed. The 2nd and 3rd harmonics of the acoustic version seemed very quiet, especially compared to my theoretical version.

This is either to do with the recording, which could have been of unsatisfactory quality or with the simplifications when creating my solutions. Firstly, we simplified our representation of a string. We ignored its rigidity, the fact that depending on the note, multiple strings could be grouped together and we only took into account movement in one dimension. Secondly, we simplified our representation of the hammer. Our initial condition was a piece-wise function, which assumes that our string was uniformly hit across a certain distance and was represented at an instant. There is also felt on the hammers, which dampens the sound. Lastly, we simplified our representation of the piano, as we only took into account the string and hammer. The soundboard, which is a large piece of wood in the piano, is responsible for a lot of the projection of the sound. The shape of which is one of the main differences between an upright and grand piano. There is also constant interaction between the strings. The frame of the piano also moves, which may even be responsible for some of the sound. In a more in-depth study, I can even imagine chaos theory being relevant.

Having spoken to some pianists who have been able to play on some of the best synthesizers, created by Roland, they always apparently lack an organic feeling. This is a perfect illustration of the difficulty of creating a synthesised electric piano, that sounds natural.

There are so many parameters that have a profound effect on the sound. If I were to extend this study, I would initially consider the felt on the hammer, then the rigidity of the string and then try to integrate the soundboard into the equation. I believe these three issues could address some of the discrepancies between the theoretical and acoustic models.

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List of figures

Appendix A

Annex

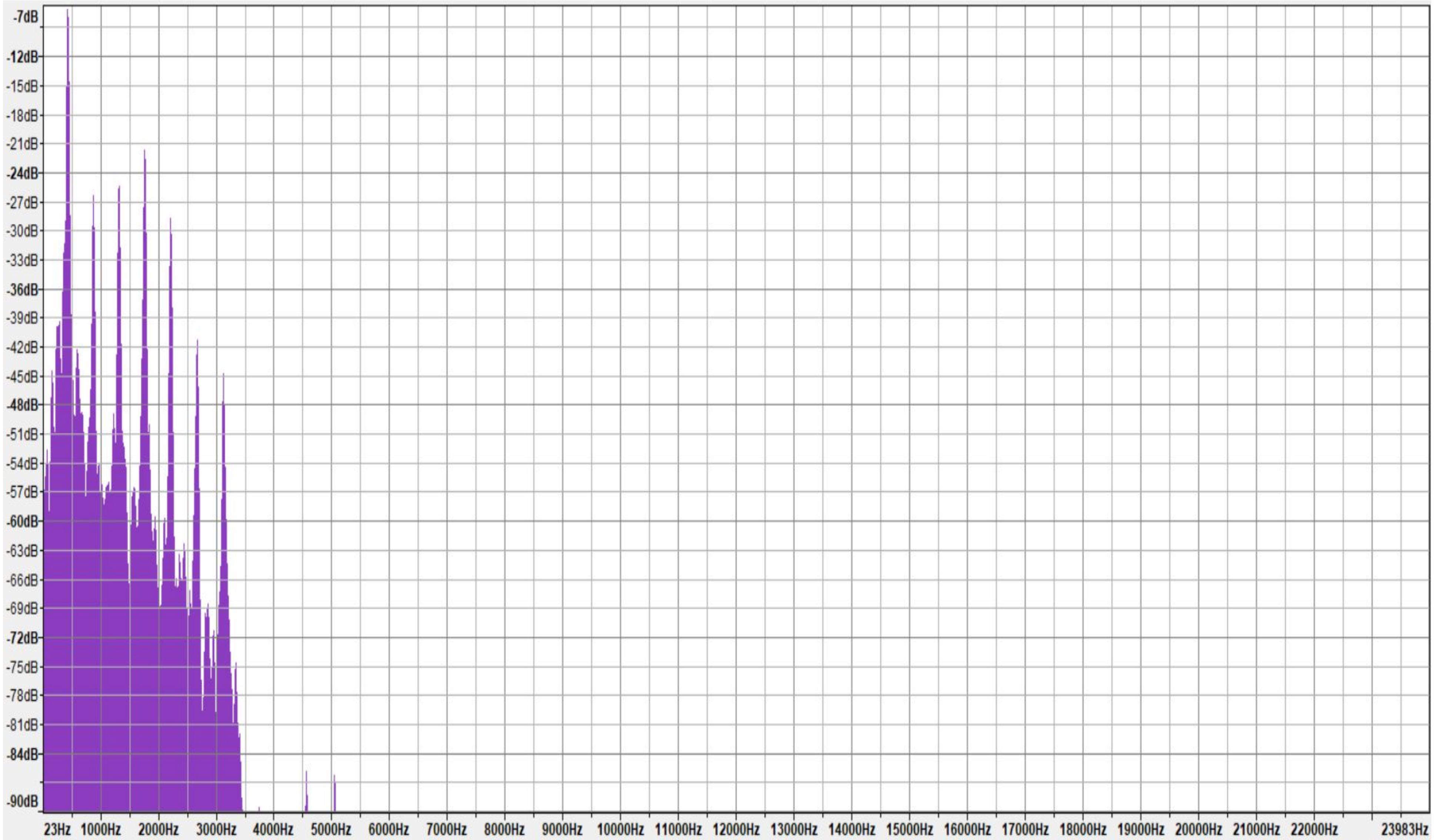

Figure A.1: Annexe A

Declaration of honour

Smith Harry
Collège du Sud
Rue de Dardens 79
1630 Bulle

Je certifie que le travail

Mathematics in Art
What happens when a hammer
hits a piano string?

été réalisé par moi conformément au "Guide de travail" des collèges et aux "Lignes directrices" de la DICS concernant la réalisation du Travail de Maturité.

Lieu, date:

Route du Prey 6 Le 20 mars 2020
Granges (Veveyse) 1614

Harry Smith